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Sharing Nonconvex Costs

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Abstract. The paper analyzes cooperative games with side payments. Each player faces a possibly non-convex optimization problem, interpreted as production planning, constrained by his resources or technology. Coalitions can aggregate (or pool) members' contributions. We discuss instances where such aggregation eliminates or reduces the lack of convexity. Core solutions are computed or approximated via dual programs associated to the grand coalition.

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1. Introduction

The general aim of this article is two-fold: first, to show that some questions related to economic cooperation may lead to familiar mathematical programming problems, and second, to review methods by which these problems can be analyzed. We thus enter an area shared by cooperative game theory and mathematical programming. We address two sorts of readers: first, mathematical economists with game-theoretic orientation who wish to construct solutions for their models (and not only prove existence theorems), and second, members of the optimization community who are curious to see the applied significance of some of their constructs.

The basic problem is described as follows. Many situations encourage the concerned parties to pool their resources, skills, or technologies so as to achieve better outcomes for everybody. Suppose here that each member i of a fixed, finite society I must produce a 'quantity' bundle q_i, belonging to a real vector space $\mathbb{E}$. If i does so in autarchy, he incurs substantial cost $c_i(q_i)$. Therefore, to reduce expenses, he might wish to coordinate his undertakings with other agents. Specifically, a coalition $S \subseteq I$ could compute the *stand-alone cost*

$$c_S(q_S) := \inf \left\{ \sum_{i \in S} c_i(x_i) \;\middle|\; \sum_{i \in S} x_i = \sum_{i \in S} q_i =: q_S \right\}, \tag{1}$$

the aim being to distribute prospective savings among its members.[1] Any voluntary arrangement should leave no subset of the contracting players worse off than alone. Therefore, a contract's viability presumes satisfaction of a vast array of participation constraints. We ask: *Can all those constraints be satisfied for the grand coalition $S = I$? And if so, how might the overall cost be shared?*

These questions fit the framework of *cooperative (coalitional) games*, featuring *side payments* and *characteristic function* $I \supseteq S \mapsto c_S(q_S) \in \mathbb{R} \cup \{+\infty\}$. For such games we let notions of efficiency and fairness be formalized by *core* solutions (axiomatized by Peleg [19]). A *cost allocation* $u = (u_i) \in \mathbb{R}^I$ belongs to the *core* iff it entails

Pareto efficiency: $\sum_{i \in I} u_i \; = \; c_I(q_I)$,
and stability; i.e., $\sum_{i \in S} u_i \; \leqslant \; c_S(q_S)$ *for all coalitions* $S \subset I$.

Stability (or coalitional rationality) means that no singleton or strict subset $S \subset I$ of several players could improve their outcome by splitting away from the society.[2] That property is easily achieved: Simply let the numbers u_i be so small that $\sum_{i \in S} u_i \leqslant c_S(q_S), \forall S \subseteq I$. So, the essential difficulty resides in the requirement that total cost be efficient and covered in full.

Shapley and Shubik (1969) studied twin versions of (1), stemming from profit sharing. Their so-called *market games* model free, frictionless exchange of goods among agents $i \in I$, having (utility or) payoff functions $\pi_i : \mathbb{E} \to \mathbb{R} \cup \{-\infty\}$ and resource endowments $q_i \in \mathbb{E}$. Then

$$\pi_S(q_S) := \sup \left\{ \sum_{i \in S} \pi_i(x_i) \;\middle|\; \sum_{i \in S} x_i = q_S \right\}, \tag{2}$$

and $u \in \mathbb{R}^I$ is declared a core allocation iff $\sum_{i \in S} u_i \geqslant \pi_S(q_S)$ for all $S \subset I$, again – of course – with equality for the grand coalition. See also [8, 10, 11].

In this paper, we reconsider such games (preferring form (1) to (2)), focusing basically on the three aspects below.

1) Mathematical programs (1) and (2) call for convexity assumptions to facilitate computation and decomposition. In economic models, convexity plays a crucial role in establishing the existence of solutions, be it core or competitive equilibria. Thus, quite understandably, mathematical programming and economic theory often assume suitable convexity, thereby excluding important cases where marginal cost (or dis-utility) may decrease. Therefore it is of importance to develop techniques for accommodating non-convexity. With this view, we use as a benchmark the fact that appropriate Lagrange multipliers, if any, generate core allocations (Theorem 1). Applications of that result rely upon convexity in the grand aggregate – a property which ensures existence of at least one multiplier. When allowing non-convex data, we estimate a number $d \geqslant 0$ that renders the said allocations stable and Pareto efficient up to deficit d (Theorem 2). Not surprisingly, as d we can use the so-called *duality gap*. We demonstrate, in the context at hand, the fundamental principle that non-convexity matters little when participants are many and minor ([2, 9, 12, 15, 23, 27]). Specifically, the Shapley-Folkman Lemma ensures that d will be relatively small in large games. The framework we deal with supplements conventional models in which the same set (and sort) of players is replicated time and again.

2) Items (1) and (2) typically embody marginal (reduced) functions c_i, π_i produced by implicit or exogenous optimization. For example, $c_i(q_i)$ could emerge as the optimal value

$$c_i(q_i) := \inf\left\{C_i \cdot y_i \;\middle|\; A_i y_i \in q_i + \mathbb{R}^m_+,\ y_i \geqslant 0\right\} \tag{3}$$

of a linear program with given $C_i \in \mathbb{R}^{n_i}$, $q_i \in \mathbb{R}^m$, and $m \times n_i$ matrix A_i.[3] In such and other cases it may facilitate modelling or computation to keep data in original, extensive form. Also, since cooperation typically requires joint effort, it may be hard – and sometimes not desirable – to fully separate concerted actions on one side from cost allocation on the other. Consequently, our second purpose is to aggregate explicit 'technological' constraints and append them to appropriate programs. Doing so we include and generalize *production games* of Owen (1975); see also [4, 5, 7, 10, 21].

3) Characteristic functions like (1) or (2) provide summary descriptions of collusive cost or worth. That summary mentions neither time nor uncertainty, features of primary importance in optimization and economics. So, a third aspect of this paper is to briefly model cooperative opportunities which unfold over time and under uncertainty. That modelling adds to the results of Sandsmark [22].

2. Cost-sharing

Suppose that no cost function c_i assumes the value $-\infty$. We emphasize though that $c_i(x_i) = +\infty$ is not excluded. In fact, we use $+\infty$ to account for violation of implicit constraints. Evidently, if some $c_i(\cdot) \equiv +\infty$, then the game has no sense. So, from here on we posit that $c_I(q_I)$ is finite, this implying that all cost functions be *proper*.[4]

Then (1) reflects exchange of perfectly divisible goods, freely transferable among members of S. The advantages of such exchange are evident; to wit, transfers (and aggregations) offer increased leeway and better substitution possibilities. Less evident is the fact that granted *convex* costs, cooperative incentives become so strong and well distributed that the grand coalition can safely form. Its formation means that costs can be shared in ways not blocked by any subgroup. This is recalled in the following, commonly known result - included for completeness:

PROPOSITION 1 (Convexity $\Rightarrow$ nonempty core). *Suppose all cost functions c_i : $E \to \mathbb{R} \cup \{+\infty\}$ are convex.*[5] *Then the cost-sharing game is balanced; i.e., it has nonempty core.*[6]

It is reassuring, of course, to know that the core associated with sharing of separable convex costs will be nonempty. We need to advance somewhat further along *two* lines though. *First*, some computational or mathematical advice must be brought forward concerning how to find core elements if any.[7] *Second*, there are good economic reasons to relax the convexity assumption in Proposition 1. We attempt to pursue both goals at once, considering a somewhat more general situation than (1) that has characteristic function

$$I \supseteq S \mapsto c_S(Q_S) := \inf \left\{ \sum_{i \in S} c_i(x_i) \;\middle|\; \sum_{i \in S} A_i x_i \in \sum_{i \in S} Q_i =: Q_S \right\}. \tag{4}$$

Here c_i maps $\mathbb{E}_i$ into $\mathbb{R} \cup \{+\infty\}$; Q_i is a nonempty subset of $\mathbb{E}$; both $\mathbb{E}_i, \mathbb{E}$ are locally convex Hausdorff vector spaces; and $A_i : \mathbb{E}_i \to \mathbb{E}$ is linear continuous.

When $\lambda : \mathbb{E} \to \mathbb{R}$ is linear continuous, we write $\lambda \in \mathbb{E}^*$ and often simply λx instead of $\lambda(x)$. The *adjoint* operator $A_i^* : \mathbb{E}^* \to \mathbb{E}_i^*$ is defined by $(A_i^*\lambda)(x_i) = \lambda(A_i x_i)$ for all $x_i \in \mathbb{E}_i$, $\lambda \in \mathbb{E}^*$. Let $f^*(\lambda) := \sup_x \{\lambda x - f(x)\}$ denote the *convex conjugate* of any given function $f : \mathbb{E} \to \mathbb{R} \cup \{\pm\infty\}$. In particular, given the extended indicator $\delta_i : \mathbb{E} \to \mathbb{R} \cup \{+\infty\}$ which equals 0 on Q_i and $+\infty$ elsewhere, its conjugate δ_i^* is called the *support function* of Q_i. Letting $x = (x_i)_{i \in I}$ we note that $L_S(x, q, \lambda) := \sum_{i \in S} \{c_i(x_i) + \lambda(q_i - A_i x_i)\}$ is a standard *Lagrangian* associated to (4). It has reduced form

$$\begin{aligned} L_S(x, \lambda) : &= \sum_{i \in S} \inf_{q_i \in Q_i} \{c_i(x_i) + \lambda(q_i - A_i x_i)\} \\ &= \sum_{i \in S} \left\{ c_i(x_i) - \lambda(A_i x_i) - \delta_i^*(-\lambda) \right\}. \end{aligned}$$

Any $\lambda_I \in \mathbb{E}^*$ such that $c_I(Q_I) \leqslant \inf_x L_I(x, \lambda_I)$ will be called a *Lagrange multiplier* (for the grand coalition).

THEOREM 1 (Lagrange multipliers yield core solutions). *Suppose $c_I(Q_I)$ is finite and that λ_I is a Lagrange multiplier for the grand coalition. Then the cost allocation*

$$u_i := -\delta_i^*(-\lambda_I) - c_i^*(A_i^* \lambda_I), \; i \in I,$$

belongs to the core.

The special instance (1) corresponds, of course, to $\mathbb{E}_i = \mathbb{E}$, $A_i = Id$, and $Q_i = \{q_i\}$. Then, if $\lambda_I \in \mathbb{E}^*$ satisfies $c_I(q_I) \leqslant \inf_x L_I(x, \lambda_I)$, the allocation $u_i := \lambda_I q_i - c_i^*(\lambda_I)$, $i \in I$, belongs to the core.

Proof. Since $\inf_{x_i \in \mathbb{E}_i} \{c_i(x_i) - \lambda_I(A_i x_i)\} = -c_i^*(A_i^* \lambda_I)$, it holds that $\inf_x L_S(x, \lambda_I) = -\sum_{i \in S} \left\{ \delta_i^*(-\lambda_I) + c_i^*(A_i^* \lambda_I) \right\}$. Stability now obtains because coalition S incurs cost

$$\begin{aligned} \sum_{i \in S} u_i = \inf_x L_S(x, \lambda_I) &\leqslant \sup_\lambda \inf_{x_i \in \mathbb{E}_i, q_i \in Q_i} L_S(x, q, \lambda) \\ &\leqslant \inf_{x_i \in \mathbb{E}_i, q_i \in Q_i} \sup_\lambda L_S(x, q, \lambda) = c_S(Q_S). \end{aligned}$$

The last inequality is often referred to as *weak duality*. The hypothesis concerning λ_I ensures *strong duality.* To wit,

$$c_I(Q_I) \leqslant \inf_x L_I(x, \lambda_I) \leqslant \sup_\lambda \inf_x L_I(x, \lambda) \leqslant \inf_{x_i \in \mathbb{E}_i, q_i \in Q_i} \sup_\lambda L_I(x, q, \lambda) = c_I(Q_I)$$

so that Pareto efficiency does indeed prevail: $c_I(Q_I) = \inf_x L_I(x, \lambda_I) = \sum_{i \in I} u_i$. □

In setting (1) Theorem 1 has a nice (and well known) interpretation: Suppose commodity bundles in $\mathbb{E}$ were traded at constant linear prices $\lambda \in \mathbb{E}^*$. Then, if i were a price-taker who could freely decide his output, he would envisage profit $c_i^*(\lambda) := \sup_{x_i} \{\lambda x_i - c_i(x_i)\}$. For arbitrary price regime λ, given already his output commitment q_i, potential profit always dominates, of course, the fait accompli, i.e., $c_i^*(\lambda) \geqslant \lambda q_i - c_i(q_i)$. Now, the particular nature of any Lagrange multiplier λ_I is that production – and profit considerations – can be decentralized: Each individual i freely maximizes his profit $x_i \mapsto \lambda_I x_i - c_i(x_i)$. The optimal choices, if any, yield $\sum_i x_i = q_I$ and $\sum_i c_i(x_i) = c_I(q_I)$. Moreover, when all c_i are differentiable, marginal cost λ_I becomes uniform across all active producers; that is, $\lambda_I = c_i'(x_i)$ for all those i. Otherwise production would be inefficient.[8] Note that with price regime λ_I in vigor, individual i is charged $\lambda_I q_i$ for the task he brings – and offered a premium $c_i^*(\lambda_I)$, representing his competitive profit contribution. In the more general setting of Theorem 1 agent i pays $\inf \lambda_I Q_i$ for production minus competitive profit $c_i^*(A_i^* \lambda_I)$ that would emerge under price $A_i^* \lambda_I$ on his inputs.

For illustration of Theorem 1, as it applies to (1), suppose that individual cost is a *marginal function*

$$c_i(q_i) := \inf \{C_i(x_i, y_i) \mid A_i x_i = q_i\}, \tag{5}$$

stemming from a bivariate proper objective C_i. In seeking a best solution agent i, if alone, might be constrained by technological bottlenecks or own resources, often scarce or available in inappropriate proportions. A coalition $S \subseteq I$ can better overcome some such hurdles and incur cost

$$c_S(q_S) := \inf_{x,y} \left\{ \sum_{i \in S} C_i(x_i, y_i) \;\middle|\; \sum_{i \in S} A_i x_i = q_S \right\}.$$

Let here $L_S(x, y, \lambda) := \sum_{i \in S} \left[C_i(x_i, y_i) + \lambda(q_i - A_i x_i)\right]$ and note that $\inf_{x,y} L_S(x, y, \lambda) = \sum_{i \in S} \left[\lambda q_i - C_i^*(A_i^* \lambda, 0)\right]$. In the proof of Theorem 1 replace x and $\inf_x$ with (x, y) and $\inf_{x,y}$, respectively to obtain:

PROPOSITION 2 (Core outcomes for inf-convolutions of marginal values). *Suppose $c_I(q_I) \leqslant \inf_{x,y} L_I(x, y, \lambda_I)$ for some Lagrange multiplier $\lambda_I \in \mathbb{E}^*$ of the grand coalition. Then, given cost functions like (5), allocation $u := [\lambda_I q_i - C_i^*(A_i^* \lambda_I, 0)]_{i \in I}$ belongs to the core.*

For additional illustration of Theorem 1 consider, once again, individual cost functions which are marginal values:

$$c_i(q_i) := \inf\{f_i(y_i) \mid g_i(y_i) \in q_i + \mathbb{K}\} \tag{6}$$

with $0 \in \mathbb{K}+\mathbb{K} \subseteq \mathbb{K} \subseteq \mathbb{E}$, and f_i, g_i mapping a space $\mathbb{E}_i$ into $\mathbb{R}\cup\{+\infty\}$ and $\mathbb{E}$ respectively. Note that (6) generalizes (3). Clearly, (6) reduces to a particular instance of (5), letting $C_i(x_i, y_i) = f_i(y_i)$ when $g_i(y_i) \in x_i + \mathbb{K}$, and $C_i(x_i, y_i) = +\infty$ otherwise. The special case (6) is, however, important enough to merit separate attention. Coalition S could now incur stand-alone cost

$$c_S(q_S) = \inf_y \left\{ \sum_{i\in S} f_i(y_i) \;\middle|\; \sum_{i\in S} g_i(y_i) \in q_S + \mathbb{K} \right\}. \tag{7}$$

Formula (7) is justified by

$$\begin{aligned} &\left\{ y \;\middle|\; \exists x : g_i(y_i) \in x_i + \mathbb{K}, \forall i \in S \;\&\; \sum_{i\in S} x_i = q_S \right\} \\ &\quad = \left\{ y \;\middle|\; \sum_{i\in S} g_i(y_i) \in q_S + \mathbb{K} \right\}. \end{aligned} \tag{8}$$

Evidently, the left-hand set in (8) is contained in the right-hand set there. For the converse inclusion single out any element y at the right, fix one member $j \in S$, and posit first $x_i := g_i(y_i)$ for all $i \in S\diagdown j$. Thereafter let $x_j := q_S - \sum_{i\in S\diagdown j} x_i$. Then $\sum_{i\in S} x_i = q_S$ and $g_i(y_i) = x_i + 0 \in x_i + \mathbb{K}$ for all $i \in S\diagdown j$. By construction and assumption there exists $k \in \mathbb{K}$ such that

$$g_j(y_j) = q_S + k - \sum_{i\in S\diagdown j} g_i(y_i) = q_S + k - \sum_{i\in S\diagdown j} x_i = x_j + k,$$

so (8) has been verified. Let here $L_S(y, \lambda) := \sum_{i\in S}\left[f_i(y_i) + \lambda(g_i(y_i) - q_i)\right]$ and note that $\inf_y L_S(y, \lambda) = -\sum_{i\in S}\left[\lambda q_i + (f_i + \lambda g_i)^*(0)\right]$. Replace x by y in the proof of Theorem 1 to have:

PROPOSITION 3 (Core solutions for inf-convoluted programs). *Suppose $c_I(q_I) \leqslant \inf_y L_I(y, \lambda_I)$ for some Lagrange multiplier λ_I belonging to*

$$\mathbb{K}^* := \left\{\lambda \in \mathbb{E}^* \mid \lambda x \leqslant 0 \text{ for all } x \in \mathbb{K}\right\}.$$

Then, given individual cost like (6), allocation $u := -\left[\lambda_I q_i + (f_i + \lambda_I g_i)^*(0)\right]_{i\in I}$ belongs to the core.

Acceptable cost sharing presumes agreement on how production should be implemented. In that regard we record:

PROPOSITION 4 (On implementation and cost attainment). *Let* $\mathbb{E}$ *be locally compact, all* c_i *be lower semicontinuous (lsc) proper, all* $A_i : \mathbb{E}_i \to \mathbb{E}$ *be continuous, and all* $Q_i = \{q_i\}$ *be singletons. Suppose*

$$\left\{ x = (x_i) \in \Pi_{i\in I}\mathbb{E}_i \;\middle|\; \sum_{i\in I} A_i x_i \in \mathcal{K},\ \sum_{i\in I} c_i(x_i) \leqslant r \right\} \text{ is compact} \tag{9}$$

for every compact $\mathcal{K}$ *and real* r*. Then* c_I *also becomes lsc proper, and the value* $c_I(q_I)$ *will be attained by some* x*. If moreover,* $\sum_{i\in I} c_i(x_i)$ *is upper semicontinuous (usc) at the said* x*, then* c_I *becomes continuous at* q_I*.*

Proof. To simplify notation – and to stress the generality of the argument – let $c(q) := \inf_x C(x, q)$ be a so-called *marginal function* defined on a locally compact space Q, with C lsc proper. If

$$\{(x, q) : q \in \mathcal{K},\ \ C(x, q) \leqslant r\} \text{ is compact}$$

for every compact $\mathcal{K}$ and real r, then straightforwardly c becomes lsc proper, and for every q there exists x such that $c(q) = C(x, q)$. Moreover, if $c(q) < +\infty$ and $C(x, q)$ is usc at q for the said x, then c becomes continuous at q. In our setting, let

$$C(x, q) := \begin{cases} \sum_{i\in I} c_i(x_i) & \text{if } \sum_{i\in I} A_i x_i = q, \\ +\infty & \text{otherwise.} \end{cases} \qquad \square$$

Suppose $c_I(q_I)$ is attained; that is, suppose $+\infty > c_I(q_I) = \sum_{i\in I} c_i(x_i)$ with $q_I = \sum_{i\in I} x_i$. Then, provided all c_i are convex, two things hold: *First*, if some function c_i is strictly convex, the corresponding component x_i becomes unique. *Second*, if all c_i are continuous at x_i, except maybe one, c_I becomes continuous whence subdifferentiable at q_I. Granted convex costs c_i, it is easy to see that λ_I is a multiplier for the grand coalition iff λ_I is a subgradient of $c_I(\cdot)$ at q_I.

3. Cooperation Over Time and Under Uncertainty

Let $\mathcal{T} := \{0, ..., T\}$ represent finitely many stages (decision epochs), from initial time 0 up to, and including, the planning horizon $T < +\infty$. (For simplicity, but with no conceptual loss, one may put $T = 1$.) Let Ω denote a probability space endowed with sigma-field $\mathcal{F}_{T+1}$ and complete probability measure P. There is given a chain $\mathcal{F}_0 \subseteq \cdots \subseteq \mathcal{F}_T \subseteq \mathcal{F}_{T+1}$ of sigma-fields, accounting for the information flow.

The space $\mathbb{E}$ consists here of mappings $\omega \mapsto x_i(\omega) = (x_{i0}, \ldots, x_{iT})(\omega)$, each part $\omega \mapsto x_{it}(\omega) \in \mathbb{R}^{m_t}$ being $\mathcal{F}_t$-measurable. Sigma-field $\mathcal{F}_t$ incorporates all relevant knowledge about the realized $\omega \in \Omega$ that is available at time t. The said measurability condition, which amounts to have $E[x_{it} | \mathcal{F}_t] = x_{it}$, reflects that each agent must at any time comply with the (same) information then available. This feature is commonly referred to as *nonanticipativity*: future knowledge cannot be exploited before it comes about.

Since diverse spaces might be considered, we shall refrain from specifying $\mathbb{E}$ further. Suffice it to say that the following decomposition property must be satisfied: For any event $B \in \mathcal{F}_{T+1}$ and elements $x, x' \in \mathbb{E}$ it must hold that $\mathbf{1}_B x + \mathbf{1}_{B^c} x' \in \mathbb{E}$.

Individual programs are here defined in the spirit of (6) as follows. For notational simplicity let $\vec{y}_{it} := (y_{i0}, \ldots, y_{it})$ where $y_{it} \in \mathbb{R}^{n_{it}}$. In state ω agent i incurs cost

$$f_i(\omega, y_i(\omega)) := \sum_{t=0}^{T} f_{it}(\omega, y_{it}(\omega)) \text{ if } g_{it}(\omega, \vec{y}_{it}(\omega)) \leqslant q_{it}(\omega) \in \mathbb{R}^{m_t} \text{ for all } t,$$

and each $\omega \mapsto y_{it}(\omega)$ is $\mathcal{F}_t$-measurable; otherwise $f_i(\omega, y_i(\omega)) = +\infty$. (As usual, inequality between random Euclidean vectors is meant to hold coordinatewise and almost surely.) Define

$$c_S(q_S) := \inf_y \left\{ \sum_{i \in S} E f_i(\omega, y_i(\omega)) \;\middle|\; \sum_{i \in S} g_{it}(\omega, \vec{y}_{it}(\omega)) \leqslant \sum_{i \in S} q_{it}(\omega) =: q_{St}(\omega) \text{ for all } t \right\}.$$

It is assumed that all the functions are jointly measurable with respect to their arguments, the expected values are well-defined, and the infima are finite. The subscript t always indicates measurability with respect to $\mathcal{F}_t$. Put

$$L_S(y, \lambda) := \sum_{i \in S} \sum_{t=0}^{T} E\left[f_{it}(\omega, y_{it}(\omega)) + \lambda_t(\omega) \{ g_{it}(\omega, \vec{y}_{it}(\omega)) - q_{it}(\omega) \} \right].$$

PROPOSITION 5 (Multistage, stochastic core elements). *Suppose $c_I(q_I) \leqslant \inf_y L_I(y, \lambda_I)$ for some $\lambda_I \in \mathbb{E}^*$. Then the cost allocation*

$$u_i := \inf_y \sum_{t=0}^{T} E\left[f_{it}(\omega, y_{it}(\omega)) + \lambda_{It}(\omega) \{ g_{it}(\omega, \vec{y}_{it}(\omega)) - q_{it}(\omega) \} \right]$$

belongs to the core. Moreover, for any interim time $t < T$, featuring sunk but optimal decisions $\vec{y}_{it}$, the remaining game with conditional cost-to-go: $c_i(\omega, q_i | \mathcal{F}_t, \vec{y}_{it}) :=$

$$\inf_{y_{i\tau}, \tau > t} \left\{ \sum_{\tau > t} E\left[f_{i\tau}(\omega, y_{i\tau}(\omega)) \,|\, \mathcal{F}_t, \vec{y}_{it} \right] : g_{i\tau}(\omega, \vec{y}_{i\tau}(\omega)) \leqslant q_{i\tau}(\omega) \text{ a.s. for all } \tau > t \right\}$$

admits a conditional core allocation

$$u_i(\omega, \mathcal{F}_t, \vec{y}_{it}) := \inf_{y_{i\tau}, \tau > t} \sum_{\tau > t} E[f_{i\tau}(\omega, y_{i\tau}(\omega)) + \lambda_{I\tau}(\omega)\{g_{i\tau}(\omega, \vec{y}_{i\tau}(\omega)) - q_{i\tau}(\omega)\} | \mathcal{F}_t, \vec{y}_{it}].$$

Proof. Only the last statement requires verification. Note that

$$c_I(q_I) = \sum_i E\left\{\sum_{\tau \leqslant t} f_{i\tau}(\omega, y_{i\tau}(\omega)) + \inf_{y_{i\tau}} \sum_{\tau > t} E\left[f_{i\tau}(\omega, y_{i\tau}(\omega)) \,| \mathcal{F}_t, \vec{y}_{it}\right]\right\}$$

where the infimum is taken over $\mathcal{F}_\tau$-measurable $y_{i\tau}$, satisfying $\sum_i g_{i\tau}(\omega, \vec{y}_{i\tau}(\omega))| \vec{y}_{it} \leqslant q_{I\tau}(\omega)$ for all $\tau > t$. Also note that

$$\inf_y L_I(y, \lambda_I) = \sum_i E\left\{\sum_{\tau \leqslant t} f_{i\tau}(\omega, y_{i\tau}(\omega)) + \lambda_{I\tau}(\omega)\{g_{i\tau}(\omega, \vec{y}_{i\tau}(\omega)) - q_{i\tau}(\omega)\} + u_i(\omega, \mathcal{F}_t, \vec{y}_{it})\right\}$$

Since $c_I(q_I) \leqslant \inf_y L_I(y, \lambda_I)$ and $\lambda_{I\tau}(\omega)\{g_{i\tau}(\omega, \vec{y}_{i\tau}(\omega)) - q_{i\tau}(\omega)\} \leqslant 0$ for all $\tau \leqslant t$, it follows that

$$\sum_i \inf_{y_{i\tau}} \sum_{\tau > t} E\left[f_{i\tau}(\omega, y_{i\tau}(\omega)) \,| \mathcal{F}_t, \vec{y}_{it}\right] \leqslant \sum_i u_i(\omega, \mathcal{F}_t, \vec{y}_{it}) \tag{10}$$

where the infimum again is taken over $\mathcal{F}_\tau$-measurable $y_{i\tau}$, satisfying

$$\sum_i g_{i\tau}(\omega, \vec{y}_{i\tau}(\omega)) \,| \vec{y}_{it} \leqslant q_{I\tau}(\omega) \text{ for all } \tau > t.$$

But the right hand member of (10) incorporates a Lagrange multiplier $(\lambda_{t+1}, \ldots, \lambda_T)$ suitable for the grand problem that remains after stage t. For that problem the conclusion now derives from Theorem 1. □

Proposition 5, in relating the core concept to subgames, inspires comparison with the non-cooperative notion of *perfectness*. There is a crucial difference though: Stressed here is only consistency over time and events. No reference is made to off-solution paths. Gale's definition of the *sequential core* fits the philosophy of Proposition 5 is [6]. His motivation was to provide a rationale for money. By contrast, money and side payments are here prerequisites for cooperation.

4. Nonconvex Cost

Theorem 1 has been central so far. It requires equality between two extremal quantities: on one side the presumably finite cost $c_I(Q_I)$, on the other side the optimal value of an associated dual program. Theory tells that such equality obtains provided $x \mapsto \sum_{i\in I} c_i(x_i)$ is convex and bounded above on a set $\mathcal{X} \subseteq \Pi_{i\in I}\mathbb{E}_i$ such that

$$0 \in int\{A\mathcal{X} - \mathcal{Q}\}. \tag{11}$$

Here $Ax := \sum_{i\in I} A_i x_i$ and $\mathcal{Q} \subseteq Q_I$. (This fact will be proven in Lemma 1 - which also makes clear that the upper-bound condition becomes superfluous when $\mathbb{E}$ is finite-dimensional.) Then $L_I(x, \lambda)$ admits a saddle value. Otherwise

$$\begin{aligned} v := c_I(Q_I) = \inf_x \sup_\lambda L_I(x,\lambda) > \sup_\lambda \inf_x L_I(x,\lambda) \\ = \sup_\lambda \sum_{i\in I} \left[-\delta_i^*(-\lambda) - c_i^*(A_i^*\lambda)\right] =: v^* \end{aligned}$$

and there is a positive so-called *duality gap* $d := v - v^*$. That gap determines how well core solutions can be approximated:

THEOREM 2 (Approximate core allocations). *Suppose (11) holds for a set $\mathcal{X}$ on which $x \mapsto \sum_{i\in I} c_i(x_i)$ is bounded above. Then there exists some $\lambda_I \in \mathbb{E}^*$ which minimizes $\sum_{i\in I}\left[\delta_i^*(-\lambda) + c_i^*(A_i^*\lambda)\right]$. Any such λ_I defines an allocation $u_i := -\delta_i^*(-\lambda_I) - c_i^*(\lambda_I)$, $i \in I$, which is stable. Moreover, it is* Pareto efficient up to deficit d *in the sense that $\sum_{i\in I} u_i = c_I(Q_I) - d$.*

Proof. The 'primal value' $c_I(Q_I)$ is finite by assumption. Let λ_I be any optimal solution to the dual problem $\sup_\lambda \sum_{i\in I}\left[-\delta_i^*(-\lambda) - c_i^*(A_i^*\lambda)\right]$. Such a solution is known to exist under constraint qualification (11). Evidently,

$$\sum_{i\in I} u_i = \sup_\lambda \sum_{i\in I}\left[-\delta_i^*(-\lambda) - c_i^*(A_i^*\lambda)\right] = c_I(Q_I) - d,$$

and $\sum_{i\in S} u_i = \sum_{i\in S}\left[-\delta_i^*(-\lambda_I) - c_i^*(A_i^*\lambda_I)\right] =$

$$\inf_x L_S(x,\lambda_I) \leqslant \sup_\lambda \inf_x L_S(x,\lambda) \leqslant \inf_x \sup_\lambda L_S(x,\lambda) = c_S(Q_S)$$

for all $S \subseteq I$. □

Theorem 2 says that if some outside benefactor would contribute d on the condition that coalition I forms, then cooperation could indeed come about. Some uncoordinated activities – say, transportation for example – may affect outside parties (or society) so adversely that efficiency gains are worth a transfer $\geqslant d$. Alternatively one might enforce Pareto efficiency and relax some coalitional constraints to get the so-called *least core*. Doing so does not fit our approach.

To appreciate or use Theorem 2, the deficit must be related to the given data. Recall that a positive d comes from lack of convexity in some domains or some functions (or both). To divorce these difficulties, we suppose first that all domains are convex. Then, following Aubin and Ekeland [1], given any $f : \mathbb{E} \to \mathbb{R} \cup \{\pm\infty\}$ with nonempty and convex *effective domain* $domf := f^{-1}(\mathbb{R})$, we measure that function's lack of convexity by the number

$$\rho(f) := \sup \left\{ f(\sum_{k \in K} \alpha_k x_k) - \sum_{k \in K} \alpha_k f(x_k) \right\},$$

the supremum being taken over all finite families $\alpha_k \geqslant 0$, $x_k \in domf$, $\sum_{k \in K} \alpha_k = 1$. Clearly, $\rho(f) \geqslant 0$, $\rho(f+g) \leqslant \rho(f) + \rho(g)$, $\rho(f) = 0 \iff f$ is convex, and the largest convex function $convf \leqslant f$ must satisfy $f - \rho(f) \leqslant convf$.

PROPOSITION 6 (Core approximation). *Let $c_I(Q_I)$ and $\sum_{i \in I} \rho(c_i)$ be finite, Q_I and all $domc_i$ be nonempty convex, and suppose (11) holds. Then a dual optimal λ_I exists, and any such λ_I defines an allocation $u_i := -\delta_i^*(-\lambda_I) - c_i^*(A_i^* \lambda_I)$, $i \in I$, which is stable and Pareto efficient up to deficit $d \leqslant \sum_{i \in I} \rho(c_i)$.*

LEMMA 1 *Suppose X and Q are convex subsets of real, locally convex, Hausdorff vector spaces $\mathcal{E}$ and $\mathbb{E}$ respectively. Let F be real-valued on $X \times Q$ and $+\infty$ elsewhere. Let $A : \mathcal{E} \to \mathbb{E}$ be linear continuous, and define $\Phi(x, q) := (F(x, q), Ax - q)$. If*

$$v := \inf \{F(x, Ax) : x \in X, Ax \in Q\}$$

and $\rho(F)$ are finite, and

$$(r^0, 0) \in int\, conv \{\Phi(X \times Q) + \mathbb{R}_{++} \times \{0\}\} \tag{12}$$

for some real number r^0, then there exists $\lambda \in \mathbb{E}^*$ such that $v \leqslant -F^*(A^*\lambda, -\lambda) + \rho(F)$. Consequently, with $v^* := \sup_\lambda \{-F^*(A^*\lambda, -\lambda)\}$ we get $0 \leqslant v - v^* \leqslant \rho(F)$.

When $\mathbb{E}$ is finite-dimensional, we may replace (12) by the weaker qualification (11). Otherwise, it suffices for (12) that F be bounded above on some set $\mathcal{X} \times \mathcal{Q} \subseteq X \times Q$ such that $0 \in int\, \{A\mathcal{X} - \mathcal{Q}\}$.

Proof of Lemma 1. A quite similar result was proven in [1] when $\mathbb{E} = \mathbb{R}^m$ and F is lsc. Their demonstration carries over almost verbatim, but is included for completeness: Let

$$r := \inf \left\{ \sum_{k \in K} \alpha_k F(x_k, q_k) : \sum_{k \in K} \alpha_k A x_k = \sum_{k \in K} \alpha_k q_k, \alpha_k > 0, \sum_{k \in K} \alpha_k = 1 \right\},$$

the infimum being taken over finite sets K. Then

$$\begin{array}{llll} (i) & r & \leqslant & v \leqslant r + \rho(F), \\ (ii) & (r, 0) & \notin & \mathcal{C} := co\Phi(X \times Q) + \mathbb{R}_{++} \times \{0\} \subset \mathbb{R} \times \mathbb{E}, \\ (iii) & \exists \lambda \in \mathbb{E}^* & \text{such that} & r \leqslant -F^*(A^*\lambda, -\lambda). \end{array}$$

The left-hand inequality in (i) derives from

$$r \leqslant \inf\{F(x,q) : x \in X,\ Ax = q \in Q\} = v,$$

and the right-hand inequality in (i) is proven as follows: Pick any $\varepsilon > 0$ and a finite family $(\alpha_k, x_k, q_k) \in \mathbb{R}_+ \times X \times Q$ with $\sum_{k\in K} \alpha_k = 1$ and $\sum_{k\in K} \alpha_k A x_k = \sum_{k\in K} \alpha_k q_k$ such that $\sum_{k\in K} \alpha_k F(x_k, q_k) \leqslant r + \varepsilon$. Define the barycenter $(\bar{x}, \bar{q}) := \sum_{k\in K} \alpha_k (x_k, q_k) \in X \times Q$ and note that

$$v - \rho(F) \leqslant F(\bar{x}, A\bar{x}) - \rho(F) \leqslant F(\bar{x}, A\bar{x}) + \sum_{k\in K} \alpha_k F(x_k, q_k) - F(\bar{x}, A\bar{x})$$
$$= \sum_{k\in K} \alpha_k F(x_k, q_k) \leqslant r + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, the desired inequality follows. In particular, (i) ensures that r is finite. Next (ii) follows directly from the definition of r, and (12) implies that the convex set $\mathcal{C}$ has nonempty interior. So, to verify (iii) separate the point $(r, 0) \in \mathbb{R} \times \mathbb{E}$ from $\mathcal{C}$ by means of a nonzero continuous linear functional $(r^*, e^*) \in \mathbb{R}^* \times \mathbb{E}^*$. Thus

$$r^* r \leqslant r^* F(x,q) + e^*(q - Ax) + r^* \mathbb{R}_+ \ \text{ for all } \ x \in X, q \in Q.$$

To have $r^* < 0$ would be impossible, and $r^* = 0$ would entail $e^* \neq 0$ & $e^*(Q - AX) \geqslant 0$, thereby contradicting that (12) $\Rightarrow$ $0 \in int(AX - Q)$. Consequently, $r^* > 0$, and letting $\lambda := e^*/r^*$ we get

$$r \leqslant \inf_{x\in X, q\in Q} \{F(x,q) + \lambda(q - Ax)\} = -F^*(A^*\lambda, -\lambda).$$

Note that for F convex, $\rho(F) = 0$ and $v^* \leqslant v = r \leqslant -F^*(A^*\lambda, -\lambda) \leqslant v^*$ so that $d = 0$ and λ would then be an optimal dual solution. □

Proof of Proposition 6. Let $\mathcal{E} := \Pi_{i\in I} \mathbb{E}_i$ and $F(x,q) := \sum_{i\in I} c_i(x_i)$ if $q \in Q := Q_I$, otherwise let $F(x,q) = +\infty$. Clearly, $X := \Pi_{i\in I} dom c_i$ and Q are both nonempty convex, and $\rho(F) \leqslant \sum_{i\in I} \rho(c_i)$. The continuous linear mapping $A : \mathcal{E} \to \mathbb{E}$ in question is $Ax := \sum_{i\in I} A_i x_i$. Its transpose $A^* : \mathbb{E}^* \to \mathcal{E}^*$ is determined by $A^*\lambda = (A_i^*\lambda)$. Qualification (11) holds and $c_I(Q_I)$ is finite. Now invoke Lemma 1 to conclude. □

In setting (1) note that

$$c_I(q_I) = \inf\left\{ r = \sum_{i\in I} r_i : c_i(x_i) \leqslant r_i, \sum_{i\in I} x_i = q_I \right\}$$
$$= \inf\left\{ r : (q_I, r) \in \sum_{i\in I} epi\, c_i \right\}$$

where for any function $f : \mathbb{E} \to \mathbb{R} \cup \{+\infty\}$ the set $epif := \{(x, r) \in \mathbb{E} \times \mathbb{R} : f(x) \leqslant r\}$ is its *epigraph*. In general, let $clconvf$ denote the largest lsc convex function $\leqslant f$. We get

$$\begin{aligned} v = c_I(q_I) &= \inf\left\{ r : (q_I, r) \in \sum_{i \in I} epic_i \right\} \\ &\geqslant \inf\left\{ r : (q_I, r) \in \sum_{i \in I} epi(clconvc_i) \right\} \\ &= \inf\left\{ \sum_{i \in I} clconvc_i(x_i) : \sum_{i \in I} x_i = q_I \right\} = v^* \end{aligned}$$

with equality if all c_i were lsc convex – or more generally, if $\sum_{i \in I} epic_i \supseteq \sum_{i \in I} epi(clconvc_i)$. It is the case that $f^* = (clconvf)^*$, so the proposed allocation $u_i = \lambda_I q_i - c_i^*(\lambda_I), i \in I$, fits the instance where $clconvc_i = c_i$. For the rest of this section let $\mathbb{E} = \mathbb{R}^m$ – as in (3). With that choice $\mathbb{E}$ Aubin and Ekeland [1] used the Shapley-Folkman Lemma to compare $\sum_{i \in I} epi(clconvc_i)$ with $\sum_{i \in I} epic_i$, and found thus an upper estimate of the duality gap. Their result yields:

PROPOSITION 7 (Core approximations with convex domains). *Let $c_I(q_I)$ be finite, every $domc_i$ be convex, and suppose (11) holds with $\mathbb{E} = \mathbb{R}^m$. Then any dual optimal $\lambda_I \in \mathbb{R}^m$ defines an allocation $u_i := \lambda_I q_i - c_i^*(\lambda_I), i \in I$, which is stable and Pareto efficient up to deficit $d \leqslant (m + 1) \max_i \rho(c_i)$.* □

Aubin and Ekeland (op. cit.) also consider instances with nonlinear A_i. We conclude this section by mentioning instances when one or more sets $domc_i$ are non-convex. Such is, for example, the case in integer programming. For simplicity we consider only format (6) with all $\mathbb{E}_i$ Euclidean spaces, $\mathbb{E} = \mathbb{R}^m$,and $\mathbb{K} = \mathbb{R}^m_-$. Suppose that $f_i : \mathbb{E}_i \to \mathbb{R} \cup \{+\infty\}$ and $g_i : \mathbb{E}_i \to \mathbb{E}$ are already defined and finite-valued on $Y_i := conv(domf_i)$. Next let $convf_i$ and $convg_i$ be the largest convex functions $\leqslant f_i$ and $\leqslant g_i$, respectively. For any $y_i \in Y_i$ define

$$\begin{aligned} \hat{f}_i(y_i) &:= \inf\left\{ f_i(\hat{y}_i) : g_i(\hat{y}_i) \leqslant convg_i(y_i) \right\} \text{ and } \hat{\rho}_i \\ &:= \sup\left\{ \hat{f}_i(y_i) - convf_i(y_i) : y_i \in Y_i \right\}. \end{aligned}$$

The following result derives from Bertsekas [3]:

PROPOSITION 8 (Core approximations for non-convex domains and functions). *Let $\mathbb{E} = \mathbb{R}^m$, $c_I(q_I)$ be finite and $\{(y_i, f_i(y_i), g_i(y_i)) : y_i \in domf_i\}$ compact for every i. Suppose that for any $y_i \in conv(domf_i)$ there exists $\hat{y}_i \in domf_i$ such that $g_i(\hat{y}_i) \leqslant convg_i(y_i)$. Then any dual optimal $\lambda_I \in \mathbb{R}^m_+$ defines an allocation $u_i := \lambda_I q_i - (f_i - \lambda_I g_i)^*(0), i \in I$, which is stable and Pareto efficient up to deficit $d \leqslant (m + 1) \max_i \hat{\rho}_i$.*

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Notes

1. By convention empty sums equal zero. It is tacitly assumed that no i misrepresents privately held information about c_i to own advantage.
2. Coalitions are here *orthogonal* in the sense that members of S can jointly incur cost $c_S(q_S)$ regardless of what the outsiders $i \in I \setminus S$ undertake. Collusion is voluntary: Outsiders can neither demand nor be forced to deal with insiders.
3. By convention inf $\emptyset = +\infty$. This instance illustrates that agents need neither share the technology nor have identical cost functions. Each party brings his own, separate production facility into the grand enterprise. Joint financing of *one common facility* (e.g. a copying machine) is not the issue here; see [16, 17, 28].
4. A hypothesis that all $c_S(q_S) \neq +\infty$ is not needed: Coalitions incurring infinite cost can safely be ignored; it suffices that $c_I(q_I)$ is finite.
5. The same *convex* $c_i = c$ yields $c_S(q_S) = |S|\, c(q_S/|S|)$ with uniform distribution of the aggregate quantity. If c also is positively 1-homogeneous, we get $c_S(q_S) = c(q_S)$.
6. Thus a cost-sharing game with convex finite-valued functions $c_S(\cdot)$ will be *totally balanced*, meaning that itself and all its subgames have nonempty cores; see [14]. Totally balanced games were first studied by Shapley and Shubik [25] who characterized them as market games with side payments. Our construction parallels theirs.
7. In general, that problem can be immense. It requires generation of $2^{\#I} - 1$ numbers $c_S(q_S)$, $\emptyset \neq S \subseteq I$, and subsequent solution of as many inequalities – say, by linear programming.
8. For generalizations to nonsmooth, convex cost see the material on *infimal convolutions* in [13] or [20].

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